distance k/h. This technique has the advantage of giving the correct result for large Fourier numbers, since for large Fourier numbers the first eigenvalue plays the dominant role in the temperature distribution.

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INDIRECT THERMAL SENSING IN COMPOSITE MEDIA

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NOMENCLATURE

$A_{ij}, A_{in}, B_{ij}, B_{in}$, constants;
C_{pi}, specific heat at constant pressure in the ith section;
D_i^2, thermal diffusivity in the ith section;
E_{in}, constant;
i, j, k, integers;
J_0, zero order Bessel function of the first kind;
K_i, thermal conductivity in the ith section;
$K_{jA}(x, t), K_{jB}(x, t), K_{jC}(x, t)$, functions of space and time, defined by equations (17)–(19);
M_{in}, N_{in}, derived eigenfunction terms (dimensionless);
Q_i, distributed source in the ith section;
t, time;
$T_i(x, t)$, temperature in the ith section;
$X_{in}(x)$, eigenfunction (dimensionless);
x, spatial coordinate;
Y_0, zero order Bessel function of the second kind;
γ_n, eigenvalue;
ρ_i, density in the ith section.

INTRODUCTION

THE PROBLEMS associated with obtaining direct experimental data in extreme environments and when space for data probes is not available has led logically to a program of indirect experimental measurements. The use of proper analytical techniques, along with indirect experimental data, allows the determination of the desired information at locations inaccessible directly by experimental probes.

In this type of problem one seeks the transient boundary conditions given the initial and some time-dependent conditions in the interior of the media. This is called an inverse problem or an interior value problem in contrast to a boundary value problem.

At present, the only technique for the solution of the inverse problem in composite media is a numerical method proposed by Beck [1]. Mulholland and San Martin [2] used the results of a known exact solution to obtain the internal and external temperature history of a composite. The objective of this paper is to present an analytical method which builds on the ideas presented in [2] for treating such problems in composite materials composed of k solidly joined plates, cylinders or spheres.

STATEMENT OF PROBLEM

The heat conduction equation for the ith section of k solidly joined plates, cylinders or spheres is

$$\frac{1}{D_i^2}\frac{\partial T_i(x,t)}{\partial t} = \nabla^2 T_i(x,t) + \frac{Q_i(x,t)}{K_i} \tag{1}$$

$$x_i \leqslant x \leqslant x_{i+1}$$

where $Q_i(x, t)$ is the rate of internal heat generation in the ith section.

The boundary, internal and initial conditions for the composite are

(a) $T_1(x_1, t) = T_1(t)$

(b) $T_k(x_{k+1}, t) = T_2(t)$

(c) $T_i(x, 0) = V_i(x),\ x_i \leqslant x \leqslant x_{i+1},\ i = 1, 2, \ldots, k$ (2)

(d) $T_i(x_{i+1}, t) = T_{i+1}(x_{i+1}, t)$

(e) $K_i \dfrac{\partial T_i(x_{i+1}, t)}{\partial x} = K_{i+1}\dfrac{\partial T_{i+1}(x_{i+1}, t)}{\partial x}$

where the functions $T_1(t)$ and $T_2(t)$ are unknown functions of time which are to be determined from known internal conditions.

SOLUTION

The solution to equations (1) and (2) has been obtained for a rectangular, cylindrical and spherical coordinate system [2, 3] and is given by

$$T_i(x,t) = \sum_{n=1}^{\infty}\Big\{g_n \exp(-\gamma_n^2 t) + \int_0^t q_n(\tau)\exp[-\gamma_n^2(t-\tau)]\,d\tau$$

$$- \sum_{j=1}^{2} l_{nj}\int_0^t \frac{dT_j(\tau)}{d\tau}\exp[-\gamma_n^2(t-\tau)]\,d\tau\Big\}X_{in}(x)$$

$$+ L_{i1}(x)\,T_1(t) + L_{i2}(x)\,T_2(t) \tag{3}$$

$$x_i \leqslant x \leqslant x_{i+1}, \quad i = 1, 2, 3, \ldots, k, \quad t \geqslant 0$$

where

$$g_n = \frac{1}{N_n}\sum_{i=1}^{k}\rho_i C_{pi}\int_{x_i}^{x_{i+1}} x^c V_i(x)\,X_{in}(x)\,dx - \sum_{j=1}^{2} l_{nj}T_j(0),$$
$$n = 1, 2, 3, \ldots \tag{4}$$

$$q_n(t) = \frac{1}{N_n}\sum_{i=1}^{k}\int_{x_i}^{x_{i+1}} x^c Q_i(x,t)\,X_{in}(x)\,dx, \quad n = 1, 2, 3, \ldots \tag{5}$$

$$l_{nj} = \frac{1}{N_n}\sum_{i=1}^{k}\rho_i C_{pi}\int_{x_i}^{x_{i+1}} x^c L_{ij}(x)\,X_{in}(x)\,dx,$$
$$n = 1, 2, 3, \ldots, j = 1, 2 \tag{6}$$

$$N_n = \sum_{i=1}^{k}\rho_i C_{pi}\int_{x_i}^{x_{i+1}} x^c [X_{in}(x)]^2\,dx, \quad n = 1, 2, 3, \ldots \tag{7}$$

$$X_{in}(x) = A_{in}M_{in}(x) + B_{in}N_{in}(x),$$
$$n = 1, 2, 3, \ldots, \quad i = 1, 2, \ldots, k \tag{8}$$

$$L_{ij}(x) = A_{ij}\,\theta(x) + B_{ij} \tag{9}$$

where the functions $M_{in}(x)$, $N_{in}(x)$ and $\theta(x)$ are given in Table 1 for plates ($c = 0$), cylinders ($c = 1$), and spheres ($c = 2$).

Table 1

Geometry	$M_{in}(x)$	$N_{in}(x)$	$\theta(x)$
Plate ($c = 0$)	$\cos\left(\frac{\gamma_n}{D_i}x\right)$	$\sin\left(\frac{\gamma_n}{D_i}x\right)$	x
Cylinder ($c = 1$)	$J_0\left(\frac{\gamma_n}{D_i}x\right)$	$Y_0\left(\frac{\gamma_n}{D_i}x\right)$	$\ln x$
Sphere ($c = 2$)	$\frac{1}{x}\sin\left(\frac{\gamma_n}{D_i}x\right)$	$\frac{1}{x}\cos\left(\frac{\gamma_n}{D_i}x\right)$	$1/x$

The coefficients A_{in}, B_{in} for $i = 1, 2, \ldots, k$, $n = 1, 2, 3, \ldots$, the coefficients A_{ij}, B_{ij} for $i = 1, 2, \ldots, k$, $j = 1, 2$ and the eigenvalues γ_n, $n = 1, 2, 3, \ldots$ which are given in equations (3)–(9) are obtained in a straightforward manner and are given in detail in [3, 4].

This solution gives the temperature distribution in any of the k plates, cylinders or spheres, that are joined at their k-1 interfaces, in terms of the two unknown temperature distributions $T_1(t)$ and $T_2(t)$; thus, the problem is now reduced to the determination of these unknown temperature distributions in terms of known internal conditions.

TEMPERATURE SPECIFIED AT TWO LOCATIONS

Consider the problem where the temperature distribution is known at two different locations within the composite. If the known temperature distributions are given by $T_P(x_{1P}, t)$, $1 \leqslant P \leqslant k$, and $T_R(x_{2R}, t)$, $1 \leqslant R \leqslant k$, where P may equal R but $x_{1P} \neq x_{2R}$ then equation (3) evaluated at x_{1P} and x_{2R} will have the following form:

$$T_P(x_{1P},t) = \sum_{n=1}^{\infty}\Big\{g_n\exp(-\gamma_n^2 t) + \int_0^t q_n(\tau)\exp[-\gamma_n^2(t-\tau)]\,d\tau$$

$$- l_{n1}\int_0^t T_1'(\tau)\exp[-\gamma_n^2(t-\tau)]\,d\tau - l_{n2}$$

$$\times \int_0^t T_2'(\tau)\exp[-\gamma_n^2(t-\tau)]\,d\tau\Big\}\{X_{Pn}(x_{1P})\}$$

$$+ L_{i1}(x_{1P})\,T_1(t) + L_{i2}(x_{2P})\,T_2(t) \tag{10}$$

and

$$T_R(x_{2R}, t) = \sum_{n=1}^{\infty} \{g_n \exp(-\gamma_n^2 t) + \int_0^t q_n(\tau) \exp[-\gamma_n^2 (t-\tau)] \, d\tau$$

$$- l_{n1} \int_0^t T'_1(\tau) \exp[-\gamma_n^2 (t-\tau)] \, d\tau - l_{n2}$$

$$\times \int_0^t T'_2(\tau) \exp[-\gamma_n^2 (t-\tau)] \, d\tau\} \{X_{Rn}(x_{2R})\}$$

$$+ L_{i1}(x_{2R}) \, T_1(t) + L_{i2}(x_{2R}) \, T_2(t). \qquad (11)$$

In equations (10) and (11), we have two equations for the two unknown quantities $T_1(t)$ and $T_2(t)$ but due to the form of these equations an approximate solution must be used. A good approximation for the surface temperature distributions can be obtained from a careful examination of the data since the time dependent portion of the solution should have the same general form for the interior as for the boundaries. This criterion is employed to select an appropriate analytical function for these distrlbutions. To illustrate, assume that the temperature distributions can be expressed as polynomials of second degree. Thus

$$T_1(t) = A_1 + B_1 t + \frac{C_1 t^2}{2} \qquad (12)$$

and

$$T_2(t) = A_2 + B_2 t + \frac{C_2 t^2}{2}. \qquad (13)$$

Substitution of equations (12) and (13) into equations (10) and (11) results in the following equations

$$T_P(x_{1P}, t) - H(x_{1P}, t) = A_1 K_{1A}(x_{1P}, t) + B_1 K_{1B}(x_{1P}, t)$$
$$+ C_1 K_{1C}(x_{1P}, t) + A_2 K_{2A}(x_{1P}, t) + B_2 K_{2B}(x_{1P}, t)$$
$$+ C_2 K_{2C}(x_{1P}, t) \qquad (14)$$

$$T_R(x_{2R}, t) - H(x_{2R}, t) = A_1 K_{1A}(x_{2R}, t) + B_1 K_{1B}(x_{2R}, t)$$
$$+ C_1 K_{1C}(x_{2R}, t) + A_2 K_{2A}(x_{2R}, t) + B_2 K_{2B}(x_{2R}, t)$$
$$+ C_2 K_{2C}(x_{2R}, t) \qquad (15)$$

where

$$H(x, t) = \sum_{n=1}^{\infty} \{g_n \exp(-\gamma_n^2 t)$$
$$+ \int_0^t q_n(\tau) \exp[-\gamma_n^2 (t-\tau)] \, d\tau\} \, X_{in}(x) \qquad (16)$$

$$K_{jA}(x, t) = L_{ij}(x), \qquad j = 1, 2 \qquad (17)$$

$$K_{jB}(x, t) = t L_{ij}(x) - \sum_{n=1}^{\infty} l_{nj} \left(\frac{1 - \exp(-\gamma_n^2 t)}{\gamma_n^2} \right)$$
$$\times X_{in}(x), j = 1, 2 \qquad (18)$$

$$K_{jC}(x, t) = \frac{t^2}{2} L_{ij}(x) - \sum_{n=1}^{\infty} l_{nj} \left(\frac{\gamma_n^2 t - 1 + \exp(-\gamma_n^2 t)}{\gamma_n^4} \right)$$
$$\times X_{in}(x), j = 1, 2 \qquad (19)$$

and where $X_{in}(x)$ is defined by equation (8).

In equations (14) and (15), the functions $H(x, t)$, $K_{jA}(x, t)$, $K_{jB}(x, t)$ and $K_{jC}(x, t)$ are known functions of the space coordinates and time while the coefficients A_1, B_1, C_1, A_2, B_2 and C_2 must be determined. The particular method for obtaining these coefficients will depend on the particular problem but as a general illustration consider the situation where the temperature history at points x_{1P} and x_{2R} are given.

If three values of time are chosen, e.g. $t = 0$, t_A and t_B, then a system of six simultaneous non-homogeneous equations will be obtained which can be solved to obtain the values of the six coefficients. A check on the accuracy of the solutions can be obtained by substituting the six coefficients into equations (14) and (15) and using these values to calculate $T_P(x_{1P}, t)$ and $T_R(x_{2R}, t)$ at values of time other than 0, t_A and t_B and then comparing these values with the known values. If greater accuracy is desired then the intervals between 0 and t_A and between t_A and t_B can be subdivided such that separate solutions are obtained for each interval and these solutions are again compared with the known values. The procedure continues in this manner until the desired accuracy is obtained.

TEMPERATURE AND HEAT FLUX SPECIFIED AT SINGLE LOCATION

For this case, it is assumed that the temperature history $T_s(x_s, t)$, $1 \leqslant s \leqslant k$ and the heat flux per unit area, $-K_s \, \partial T_s(x_s, t)/\partial x$, are known at some position x_s in the sth layer. Under these conditions, equation (3) and its derivative will have the following forms:

$$T_s(x_s, t) = H(x_s, t) + L_{i1}(x_s) \, T_1(t) + L_{i2}(x_s) \, T_2(t)$$
$$- \sum_{n=1}^{\infty} l_{n1} \{\int_0^t T'_1(\tau) \exp[-\gamma_n^2 (t-\tau)] \, d\tau\} \, X_{sn}(x_{sn})$$
$$- \sum_{n=1}^{\infty} l_{n2} \{\int_0^t T'_2(\tau) \exp[-\gamma_n^2 (t-\tau)] \, d\tau\} \, X_{sn}(x_{sn}) \qquad (20)$$

$$\frac{\partial T_s(x_s, t)}{\partial x} = \frac{-\ddot{q}}{K_s} = \frac{\partial H(x_s, t)}{\partial x} + \frac{dL_{i1}(x_s)}{dx} T_1(t)$$
$$+ \frac{dL_{i2}(x)}{dx} T_2(t) - \sum_{n=1}^{\infty} l_{n1} \{\int_0^t T'_1(\tau) \exp[-\gamma_n^2 (t-\tau)] \, d\tau\}$$
$$\times \frac{dX_{sn}(x_{sn})}{dx} - \sum_{n=1}^{\infty} l_{n2} \{\int_0^t T'_2(\tau) \exp[-\gamma_n^2 (t-\tau)] \, d\tau\}$$
$$\times \frac{dX_{sn}(x_{sn})}{dx}. \qquad (21)$$

The equations (20) and (21) can now be solved for the unknown temperature distribution in the same manner as before. The results when substituted into equation (3) will give the temperature distribution at any position within the composite.

EXAMPLE

Consider a two layered composite composed of 40 per cent nickel steel and copper. The property values for each layer are

$0 \leqslant x \leqslant 15{\cdot}2\,\text{cm}$	$15{\cdot}2 \leqslant x \leqslant 30{\cdot}5\,\text{cm}$
$\rho_1 = 8169{\cdot}5\,\text{kg/m}^3$	$\rho_2 = 8954{\cdot}4\,\text{kg/m}^3$
$K_1 = 10{\cdot}38\,\text{W/mK}$	$K_2 = 385{\cdot}79\,\text{W/mK}$
$C_{p1} = 460{\cdot}46\,\text{J/kg K}$	$C_{p2} = 383{\cdot}02\,\text{J/kg K}$
$D_1^2 = 0{\cdot}01\,\text{m}^2/\text{h}$	$D_2^2 = 0{\cdot}404\,\text{m}^2/\text{h}.$

It will be assumed that the initial temperature distribution is known to be zero while the temperature distrlbution at $x = 12{\cdot}2$ cm and $x = 18{\cdot}3$ cm is known. Proceeding in the same manner as outlined previously, we obtain

$$X_{1n}(x) = \sin(10{\cdot}0\,\gamma_n x)$$

$$X_{2n}(x) = A_{2n}\cos(1{\cdot}57\,\gamma_n x) + B_{2n}\sin(1{\cdot}57\,\gamma_n x)$$

$$A_{2n} = \frac{\sin(1{\cdot}52\,\gamma_n)\sin(0{\cdot}478\,\gamma_n)}{\sin(0{\cdot}239\,\gamma_n)}$$

$$B_{2n} = \frac{\sin(1{\cdot}52\,\gamma_n)\cos{\cdot}0{\cdot}478\,\gamma_n)}{\sin(0{\cdot}239\,\gamma_n)}$$

$$L_{11}(x) = 1 - 0{\cdot}064\,x, \qquad L_{12}(x) = 0{\cdot}064\,x$$

$$L_{21}(x) = 0{\cdot}00172(30{\cdot}5 - x), \qquad L_{22}(x) = 1 + 0{\cdot}00172(x - 30{\cdot}5)$$

$$g_n = 0, \qquad q_n = 0$$

$$l_{n1} = \frac{18{\cdot}457\sin(0{\cdot}239\,\gamma_n)}{14{\cdot}025\,\gamma_n\sin(0{\cdot}239\,\gamma_n) + 12{\cdot}7875\,\gamma_n\sin(1{\cdot}52\,\gamma_n)}$$

and

$$l_{n2} = \frac{106{\cdot}9\sin(1{\cdot}52\,\gamma_n)\sin(0{\cdot}239\,\gamma_n)}{14{\cdot}025\,\gamma_n\sin^2(0{\cdot}239\,\gamma_n) + 12{\cdot}7875\,\gamma_n\sin^2(1{\cdot}52\,\gamma_n)}.$$

Equations (22) and (23) become

$$\begin{aligned} T_P(12{\cdot}2, t) &= 0{\cdot}221\,A_1 + B_1[0{\cdot}221\,t \\ &- \sum_{n=1}^{\infty} l_{n1}E_{1n}X_{1n}(12{\cdot}2)] + C_1[0{\cdot}1105\,t^2 \\ &- \sum_{n=1}^{\infty} l_{n1}E_{2n}X_{1n}(12{\cdot}2)] + 0{\cdot}779\,\text{A}_2 \\ &+ B_2[0{\cdot}779\,t - \sum_{n=1}^{\infty} l_{n2}E_{1n}X_{1n}(12{\cdot}2)] \\ &+ C_2[0{\cdot}3895\,t^2 - \sum_{n=1}^{\infty} l_{n2}E_{2n}X_{1n}(12{\cdot}2)] \end{aligned} \tag{22}$$

and

$$\begin{aligned} T_R(18{\cdot}3, t) &= 0{\cdot}021\,A_1 + B[0{\cdot}021\,t \\ &- \sum_{n=1}^{\infty} l_{n1}E_{1n}X_{2n}(18{\cdot}3)] + C_1[0{\cdot}01\,t^2 \\ &- \sum_{n=1}^{\infty} l_{n1}E_{2n}X_{2n}(18{\cdot}3)] + 0{\cdot}979\,A_2 \\ &+ B_2[0{\cdot}979\,t - \sum_{n=1}^{\infty} l_{n2}E_{1n}X_{2n}(18{\cdot}3)] \\ &+ C_2[0{\cdot}4895\,t^2 - \sum_{n=1}^{\infty} l_{n2}E_{2n}X_{2n}(18{\cdot}3)] \end{aligned} \tag{23}$$

where

$$X_{1n}(12{\cdot}2) = \sin(1{\cdot}216\,\gamma_n)$$

$$X_{2n}(18{\cdot}3) = A_{2n}\cos(0{\cdot}287\,\gamma_n) + B_{2n}\sin(0{\cdot}287\,\gamma_n)$$

$$E_{1n} = \frac{1 - \exp(-\gamma_n^2 t)}{\gamma_n^2}$$

$$E_{2n} = \frac{\gamma_n^2 t - 1 + \exp(-\gamma_n^2 t)}{\gamma_n^4}.$$

When the known values of $T_P(12{\cdot}2, t)$ and $T_R(17{\cdot}3, t)$ are substituted into equations (22) and (23) and the previously mentioned procedure is employed, the values of the coefficients A_1, B_1, C_1, A_2, B_2, C_2 are obtained. Figure 1 compares the exact values of the temperature distribution) at the boundaries and the values obtained by means of the approximation.

Substitution of the known coefficients into equations (12) and (13) and then using these results in equation (3) gives the equation for the temperature within the composite as a function of position and time. This equation is

$$\begin{aligned} T_i(x, t) &= \left[A_1 + B_1 t + \frac{C_1 t^2}{2}\right]L_{i1}(x) \\ &+ \left[A_2 + B_2 t + \frac{C_2}{2}t^2\right]L_{i2}(x) \\ &- \sum_{n=1}^{\infty}\left\{\sum_{j=1}^{2} l_{nj}B_j\left[\frac{1 - \exp(-\gamma_n^2 t)}{\gamma_n^2}\right] + \frac{C_j}{2}\right. \end{aligned}$$

$$\times \left[\frac{\gamma_n^2 - 1 + \exp(-\gamma_n^2 t)}{\gamma_n^4} \right] \Bigg\} X_{in}(x). \quad (24)$$

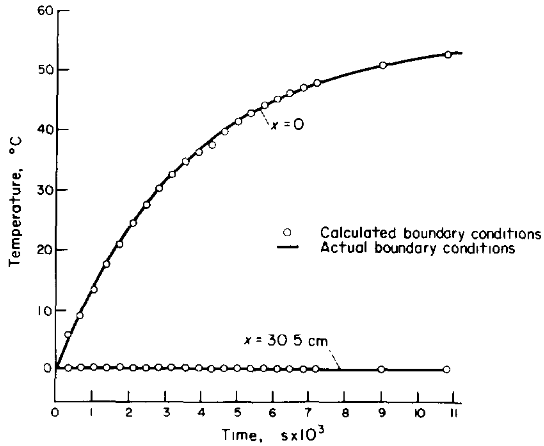

FIG. 1. Surface temperatures.

SUMMARY AND CONCLUSIONS

A method is presented which enables one to predict the internal and external thermal history of a composite composed of k discrete layers from specified internal conditions. The internal values can be either a temperature distribution or a temperature gradient and can be specified at separate points or at the same location within the solid. The accuracy of the analysis technique is well illustrated by the example where comparisons are made between the analytically generated experimental data and the temperatures obtained using the present method.

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